The Finite Satisfiability Problem for PCTL is Undecidable

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The Satisfiability Problem

- Given a formula φ of some logic $\mathcal{L}$, does φ have a model?
- Closely related to the validity problem: φ is not satisfiable iff $\neg\varphi$ is a tautology.

Satisfiability and Validity for First-Order Logic

- The decidability of validity (Entscheidungsproblem) was questioned by Hilbert and Ackermann in 1928.
- By Gödel's completeness theorem, validity is equivalent to provability, and hence semi-decidable.
- Validity is not decidable (Turing, 1936). Hence, satisfiability is not even semi-decidable.
- Finite satisfiability is undecidable (Trakhtenbrot, 1950).
- Consequently, finite validity is not semi-decidable, and hence there is no deductive system satisfying $\models_f \varphi$ iff $\vdash \varphi$.

Satisfiability for Probabilistic Temporal Logics (1)

- Probabilistic CTL (Hansson & Jonsson, 1994):

$$\begin{array}{lcl} \varphi & ::= & a \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid P(\Phi) \bowtie r \\ \Phi & ::= & \mathbf{X}\,\varphi \mid \varphi_1\,\mathbf{U}\,\varphi_2 \mid \varphi_1\,\mathbf{U}^k\,\varphi_2 \end{array}$$

  Here, $a \in AP$, $\bowtie \in \{\geq, >, \leq, <, =\}$, $r \in [0,1]$ is a rational constant, and $k \in \mathbb{N}$.
- $\mathbf{F}\,\varphi$ and $\mathbf{F}^k\,\varphi$ abbreviate $\mathit{true}\,\mathbf{U}\,\varphi$ and $\mathit{true}\,\mathbf{U}^k\,\varphi$.
- We write $\mathbf{X}_{=1}\,\varphi$ instead of $P(\mathbf{X}\,\varphi) = 1$, $\mathbf{G}_{=1}\,\varphi$ instead of $\mathbf{F}_{=0}\,\neg\varphi$, etc.
- The qualitative fragment of PCTL is obtained by restricting r to 0 and 1.
  - $\mathbf{G}_{=1}(\mathit{request} \Rightarrow \mathbf{F}_{=1}\,\mathit{granted})$
- PCTL formulae are interpreted over Markov chains.

Satisfiability for Probabilistic Temporal Logics (2)

PCTL does not have the small model property. There are (even qualitive) satisfiable PCTL formulae with only infinite-state models. Therefore, the PCTL satisfiability problem has been studied in two basic variants:

- general satisfiability, i.e., the existence of an unrestricted model;
- finite satisfiability, i.e., the existence of a finite-state model.

Positive decidability results exist for various PCTL fragments (in particular, for qualitative PCTL). The unrestricted PCTL, the general/finite satisfiability remained open for more 30 years.

Finite Satisfiability for Unrestricted PCTL

Observation 1

For a given PCTL formula φ and a given $n \geq 1$, the existence of model for φ with at most n states is decidable by encoding the question in first-order arithmetic of the reals.

Consequently,

- the finite PCTL satisfiability is semidecidable;
- if there is a computable function $f : \mathbb{N} \to \mathbb{N}$ such that every finite satisfiable PCTL formula φ has a model with at most $f(|\varphi|)$ states, then the finite PCTL satisfiability is decidable;
- the existence of f is plausible (how about the Ackermann function. . . ?)

Finite Satisfiability for Unrestricted PCTL (2)

Theorem 2 (Chodil, K., LICS 2024; JACM 2026)

The finite PCTL satisfiability is undecidable. Consequently, there is no complete deductive system proving all finitely valid PCTL formulae.

- The undecidability result holds even for PCTL formulae of the form

  $$\varphi \wedge \mathbf{G}_{=1}\, \psi$$

  where φ, ψ use only the temporal connectives $\mathbf{X}$ and $\mathbf{F}^2$.
- There is no computable upper bound on the size of a model for a given finite satisfiable PCTL formula.
- The proof requires novel&tricky techniques.

General Satisfiability for Unrestricted PCTL

Theorem 3 (Chodil, K., 2025)

The general PCTL satisfiability is highly undecidable, i.e., beyond the arithmetical hierarchy. The same holds for the PCTL validity problem.